

A MINIMAL INTEGRAL OF THE RIEMANN  $\Xi$ -FUNCTION

JAN MOSER

ABSTRACT. In this paper we obtain an equilibrium sequence  $\{\omega_n\}$  for which the following holds true: the areas (measures) of the figures corresponding to the positive and negative parts, respectively, of the graph of the function  $\Xi(t)$ ,  $t \in [\omega_n, \omega_{n+1}]$  are equal.

Dedicated to the 500th anniversary of rabbi Löw.

## 1. THE RESULT

1.1. Hardy proved in 1914 the following fundamental theorem: the function  $\zeta\left(\frac{1}{2} + it\right)$  has infinitely many real zeros (see [1]). To prove this Hardy used the following complicated formula

$$(1.1) \quad \int_0^\infty \frac{\Xi(t)}{t^2 + \frac{1}{4}} t^{2n} \cosh(\mu t) dt \sim (-1)^n \frac{\pi}{2^{2n}} \cos \frac{\pi}{8}, \quad \mu \rightarrow \frac{\pi}{4} - 0$$

where

$$\Xi(z) = \xi\left(\frac{1}{2} + iz\right), \quad \xi(s) = \frac{s(s-1)}{2} \pi^{-s/2} \Gamma\left(\frac{s}{2}\right) \zeta(s), \quad s = \sigma + it.$$

Let us note that (1.1) follows from one well-known Ramanujan's formula (see [5], p. 36).

In this direction we obtain the following

**Theorem.** There is an increasing sequence  $\{\omega_n\}_{n=1}^\infty$  such that

$$(1.2) \quad \int_{\omega_n}^\infty \Xi(t) dt = 0, \quad n = 1, 2, \dots,$$

and

$$(1.3) \quad \omega_{n+1} - \omega_n < \omega_n^{\frac{1}{6} + \epsilon}.$$

*Remark 1.* The integral (1.2) is to be named *the minimal integral* of the Riemann  $\Xi(t)$ -function (comp. (1.1)).

1.2. From (1.2) follows immediately

**Corollary 1.**

$$(1.4) \quad \int_{[\omega_n, \infty)^+} \Xi(t) dt = - \int_{[\omega_n, \infty)^-} \Xi(t) dt, \quad n = 1, 2, \dots$$

where

$$[\omega_n, \infty)^+ = \{t : \Xi(t) > 0, t \in [\omega_n, \infty)\}, \dots$$

---

*Key words and phrases.* Riemann zeta-function.

*Remark 2.* The global law of the exact equality of the areas (measures) of the figures corresponding (by the usual way) to the positive and negative parts, respectively, of the graph of the function  $\Xi(t)$ ,  $t \in [\omega_n, \infty)$  is expressed by the formula (1.4).

Next, from (1.2) we obtain

**Corollary 2.**

$$(1.5) \quad \int_{\omega_n}^{\omega_{n+1}} \Xi(t) dt = 0, \quad n = 1, 2, \dots,$$

and consequently

$$(1.6) \quad \int_{[\omega_n, \omega_{n+1}]^+} \Xi(t) dt = - \int_{[\omega_n, \omega_{n+1}]^-} \Xi(t) dt.$$

*Remark 3.* The local law of the exact equality of the areas (measures) of the figures corresponding (by the usual way) to the positive and negative parts, respectively, of the graph of the function  $\Xi(t)$ ,  $t \in [\omega_n, \omega_{n+1}]$  is expressed by the formula (1.6).

We obtain, by making use of the mean-value theorem in the formula (1.5),

$$\Xi(c) = 0, \quad c \in (\omega_n, \omega_{n+1}) \Rightarrow c = \gamma : \zeta\left(\frac{1}{2} + i\gamma\right) = 0,$$

i.e. we have

**Corollary 3.** The following canonical property

$$(1.7) \quad \{c\}_n = \{\gamma\}_n, \quad \{c\}_n \subset (\omega_n, \omega_{n+1}),$$

holds true, i.e. the set  $\{c\}_n$  of the mean-value-points of the function  $\Xi(t)$ ,  $t \in [\omega_n, \omega_{n+1}]$  is identical with the set of zeros  $\{\gamma\}_n$  of this function ( $\Xi(\gamma) = 0 \Leftrightarrow \zeta(\frac{1}{2} + i\gamma) = 0$ ).

*Remark 4.* By means of (1.4), (1.6), (1.7) we have named the sequence  $\{\omega_n\}_n$  as *the equilibrium sequence*.

*Remark 5.* Let us remind explicitly that the proof of the minimal integral (1.2) is, at the same time, the new kind of the proof of the mentioned Hardy's theorem (comp. (1.7)).

*Remark 6.* The small improvements of the exponent  $\frac{1}{6}$  in (1.3) are irrelevant. In this direction, see our discussion connected with the I.M. Vinogradov's scepticism on possibilities of the method of trigonometric sums (see [4]).

## 2. THE MAIN FORMULA

2.1. Let us remind the Riemann-Siegel formula

$$(2.1) \quad Z(t) = 2 \sum_{n \leq x(t)} \frac{1}{\sqrt{n}} \cos\{\vartheta(t) - t \ln n\} + \mathcal{O}(t^{-1/4}), \quad x(t) = \sqrt{\frac{t}{2\pi}},$$

(see [5], pp. 79. 221) where

$$(2.2) \quad \begin{aligned} Z(t) &= e^{i\vartheta(t)} \zeta\left(\frac{1}{2} + it\right), \quad \vartheta(t) = -\frac{1}{2}t \ln \pi + \operatorname{Im} \left\{ \ln \Gamma\left(\frac{1}{4} + i\frac{t}{2}\right) \right\} = \\ &= \frac{1}{2}t \ln \frac{t}{2\pi} - \frac{1}{2}t - \frac{1}{8}\pi + \mathcal{O}(t^{-1}), \quad \vartheta'(t) = \frac{1}{2} \ln \frac{t}{2\pi} + \mathcal{O}(t^{-1}), \quad \vartheta''(t) \sim \frac{1}{2t}, \end{aligned}$$

and the Gram's sequence  $\{t_\nu\}$  is defined by the equation (see [5])

$$(2.3) \quad \vartheta(t_\nu) = \pi\nu, \quad \nu \geq \nu_0 \geq 1.$$

2.2. Let

$$(2.4) \quad \begin{aligned} \Phi_1(T) &= \int_T^\infty e^{-\alpha t} t^\beta Z(t) dt = \lim_{T' \rightarrow \infty} \int_T^{T'} e^{-\alpha t} t^\beta Z(t) dt = \lim_{T' \rightarrow \infty} S(T, T'), \\ \alpha &= \frac{\pi}{4}, \quad \beta = \frac{7}{4}, \quad Z(t) = \mathcal{O}(t^{1/4}), \quad T \geq T_0 > 0, \end{aligned}$$

where  $T_0$  is a sufficiently big. By the formula (2.1) we have

$$(2.5) \quad S(T, T') = 2S_1 + 2S_2 + 2S_3 + Q$$

where

$$(2.6) \quad \begin{aligned} S_1 &= \int_T^{T'} e^{-\alpha t} t^\beta \cos\{\vartheta(t)\} dt, \quad Q = \mathcal{O}\left(\int_T^\infty e^{-\alpha t} t^{\beta-\frac{1}{4}} dt\right), \\ S_2 &= \sum_{2 \leq n \leq \sqrt{\frac{T'}{2\pi}}} \frac{1}{\sqrt{n}} \int_T^{T'} e^{-\alpha t} t^\beta \cos\{\vartheta(t) - t \ln n\} dt, \\ S_3 &= \sum_{\sqrt{\frac{T}{2\pi}} < n \leq \sqrt{\frac{T'}{2\pi}}} \frac{1}{\sqrt{n}} \int_{T_1=2\pi n^2}^{T'} e^{-\alpha t} t^\beta \cos\{\vartheta(t) - t \ln n\} dt \end{aligned}$$

by using the formula

$$\int_T^{T'} \sum_{n \leq \sqrt{\frac{t}{2\pi}}} = \sum_{n \leq \sqrt{\frac{T'}{2\pi}}} \int_{T_1}^{T'}, \quad T_1 = \max\{T, 2\pi n^2\}.$$

2.3. We obtain the following formula for the integral in  $S_2$

$$(2.7) \quad \begin{aligned} &\int_T^{T'} e^{-\alpha t} t^\beta \cos\{\vartheta(t) - t \ln n\} dt = \frac{1}{\alpha} e^{-\alpha T} T^\beta \cos\{\vartheta(T) - T \ln n\} - \\ &- \frac{1}{\alpha^2} e^{-\alpha T} T^\beta \{\vartheta'(T) - \ln n\} \sin\{\vartheta(T) - T \ln n\} - \\ &- \frac{1}{\alpha^2} \int_T^{T'} e^{-\alpha t} t^\beta \{\vartheta'(t) - \ln n\}^2 \cos\{\vartheta(t) - t \ln n\} dt - \\ &- \frac{2\beta}{\alpha^2} \int_T^{T'} e^{-\alpha t} t^{\beta-1} \{\vartheta'(t) - \ln n\} \sin\{\vartheta(t) - t \ln n\} dt - \\ &- \frac{1}{\alpha^2} \int_T^{T'} e^{-\alpha t} t^\beta \vartheta''(t) \sin\{\vartheta(t) - t \ln n\} dt + \\ &+ \frac{\beta(\beta-1)}{\alpha^2} \int_T^{T'} e^{-\alpha t} t^{\beta-1} \cos\{\vartheta(t) - t \ln n\} dt + \\ &+ \mathcal{O}(e^{-\alpha T} T^{\beta-1}) + \mathcal{O}(e^{-\alpha T'} (T')^\beta). \end{aligned}$$

2.4. Since (see (2.2))

$$(2.8) \quad \{\vartheta'(t) - \ln n\}^2 = \{\vartheta'(T) - \ln n\}^2 + \mathcal{O}\left\{(t-T)\frac{\ln T}{T}\right\}; \quad \frac{\ln x}{x} < \frac{\ln T}{T}, \quad x > T,$$

and

$$(2.9) \quad \begin{aligned} & \int_T^{T'} e^{-\alpha t} t^\beta (t-T) dt = \\ & = -\frac{1}{\alpha} e^{-\alpha T'} T'^\beta (T' - T) + \frac{\beta+1}{\alpha} \int_T^{T'} e^{-\alpha t} t^\beta dt - \frac{\beta}{\alpha} \int_T^{T'} e^{-\alpha t} t^{\beta-1} dt = \\ & \dots = \mathcal{O}\{e^{-\alpha T'} T'^{\beta+1}\} + \mathcal{O}(e^{-\alpha T} T^\beta) + \mathcal{O}\left\{\int_T^{T'} e^{-\alpha t} t^{\beta-2} dt\right\} = \\ & = \mathcal{O}\{e^{-\alpha T'} T'^{\beta+1}\} + \mathcal{O}(e^{-\alpha T} T^\beta); \quad \beta - 2 < 0, \end{aligned}$$

then  $(\frac{\ln T}{T} < 1, T > e)$

$$(2.10) \quad \frac{\ln T}{T} \int_T^{T'} e^{-\alpha t} t^\beta (t-T) dt = \mathcal{O}\left(e^{-\alpha T} T^\beta \frac{\ln T}{T}\right) + \mathcal{O}\{e^{-\alpha T'} T'^{\beta+1}\}.$$

Hence, for the first integral on the right-hand side of (2.7) we obtain (see (2.8), (2.10))

$$(2.11) \quad \begin{aligned} & \int_T^{T'} e^{-\alpha t} t^\beta \{\vartheta'(t) - \ln n\}^2 \cos\{\vartheta(t) - t \ln n\} dt = \\ & = \{\vartheta'(T) - \ln n\}^2 \int_T^{T'} e^{-\alpha t} t^\beta \cos\{\vartheta(t) - t \ln n\} dt + \mathcal{O}\left(e^{-\alpha T} T^\beta \frac{\ln T}{T}\right) + \\ & + \mathcal{O}\{e^{-\alpha T'} T'^{\beta+1}\}. \end{aligned}$$

2.5. For all the remaining integrals we obtain by the same method the estimate

$$(2.12) \quad \mathcal{O}(e^{-\alpha T} T^{\beta-1} \ln T).$$

Thus, from (2.7) by (2.11), (2.12) we obtain the following formula (see  $S_2$  in (2.6))

$$(2.13) \quad \begin{aligned} & \int_T^{T'} e^{-\alpha t} t^\beta \cos\{\vartheta(t) - t \ln n\} dt = \\ & = \frac{1}{\alpha} \frac{e^{-\alpha T} T^\beta}{1 + \frac{1}{\alpha^2} \{\vartheta'(T) - \ln n\}^2} \left[ \cos\{\vartheta(T) - T \ln n\} - \right. \\ & \left. - \frac{1}{\alpha} \{\vartheta'(T) - \ln n\} \sin\{\vartheta(T) - T \ln n\} \right] + \mathcal{O}\left(e^{-\alpha T} T^\beta \frac{\ln T}{T}\right) + \\ & + \mathcal{O}\{e^{-\alpha T'} T'^{\beta+1}\}, \end{aligned}$$

and consequently, for the integral  $S_1$  (see (2.6)) we have

$$(2.14) \quad \begin{aligned} & \int_T^{T'} e^{-\alpha t} t^\beta \cos\{\vartheta(t)\} dt = \frac{1}{\alpha} \frac{e^{-\alpha T} T^\beta}{1 + \frac{1}{\alpha^2} \{\vartheta'(T)\}^2} \left[ \cos\{\vartheta(T)\} - \right. \\ & \left. - \frac{1}{\alpha} \vartheta'(T) \sin\{\vartheta(T)\} \right] + \mathcal{O}\left(e^{-\alpha T} T^\beta \frac{\ln T}{T}\right) + \mathcal{O}\{e^{-\alpha T'} T'^{\beta+1}\}. \end{aligned}$$

2.6. Since for the integral in  $S_3$  (see (2.6), comp. (2.9)) the estimate

$$\int_{T_1}^{T'} e^{-\alpha t} t^\beta \cos\{\vartheta(t) - t \ln n\} dt = \mathcal{O}(e^{-\alpha T_1} T_1^\beta) = \mathcal{O}(e^{-2\pi\alpha n^2} n^{2\beta})$$

holds true, then

$$(2.15) \quad e^{\alpha T} T^{-\beta} S_3 = \mathcal{O} \left\{ \sum_{\sqrt{\frac{T}{2\pi}} < n \leq \sqrt{\frac{T'}{2\pi}}} \frac{1}{\sqrt{n}} \left( \frac{n^2}{T} \right)^\beta e^{\alpha(-2\pi n^2 + T)} \right\}.$$

Putting in (2.15)  $n = n_0 + k$  where

$$n_0 - 1 \leq \sqrt{\frac{T}{2\pi}} < n_0; \quad -2\pi n_0^2 + T < 0$$

we continue to manipulate with this estimate by the following way

$$(2.16) \quad \begin{aligned} &= \mathcal{O} \left\{ \sum_{\sqrt{\frac{T}{2\pi}} < n_0 + k \leq \sqrt{\frac{T'}{2\pi}}} \frac{1}{\sqrt{n_0 + k}} \left( \frac{n_0 + k}{n_0 - 1} \right)^{2\beta} e^{\alpha(-2\pi(n_0 + k)^2 + T)} \right\} = \\ &= \mathcal{O} \left\{ \frac{1}{\sqrt{n_0}} \left[ \left( \frac{n_0}{n_0 - 1} \right)^{2\beta} + \sum_{k=1}^{\infty} \left( \frac{n_0 + k}{n_0 - 1} \right)^{2\beta} e^{-2\pi\alpha(k^2 + 2n_0 k)} \right] \right\} = \\ &= \mathcal{O} \left( \frac{1}{\sqrt{n_0}} \right) = \mathcal{O}(T^{-1/4}) \Rightarrow S_3 = \mathcal{O}(e^{-\alpha T} T^{\beta - \frac{1}{4}}). \end{aligned}$$

For the remainder  $Q$  (see (2.6)) we obtain by the usual way (see (2.9)) the estimate

$$(2.17) \quad Q = \mathcal{O}(e^{-\alpha T} T^{\beta - \frac{1}{4}}).$$

2.7. Hence, from (2.4) by (2.6), (2.7), (2.13)-(2.17) we obtain the following

**Lemma 1.**

$$(2.18) \quad \begin{aligned} \Phi_1(T) &= \int_T^\infty e^{-\alpha t} t^\beta Z(t) dt = \\ &= \frac{2}{\alpha} \frac{e^{-\alpha T} T^\beta}{1 + \frac{1}{\alpha^2} \{\vartheta'(T)\}^2} \left[ \cos\{\vartheta(T)\} - \frac{1}{\alpha} \vartheta'(T) \sin\{\vartheta(T)\} \right] + \\ &+ \frac{2}{\alpha} \sum_{2 \leq n \leq \sqrt{\frac{T}{2\pi}}} \frac{e^{-\alpha T} T^\beta}{\sqrt{n} (1 + \frac{1}{\alpha^2} \{\vartheta'(T) - \ln n\}^2)} \left[ \cos\{\vartheta(T) - T \ln n\} - \right. \\ &\left. - \frac{1}{\alpha} \{\vartheta'(T) - \ln n\} \sin\{\vartheta(T) - T \ln n\} \right] + \mathcal{O}(e^{-\alpha T} T^{\beta - \frac{1}{4}}), \end{aligned}$$

for all sufficiently big  $T > 0$  and  $\alpha = \frac{\pi}{4}$ ,  $\beta = \frac{7}{4}$ .

### 3. THE FORMULA FOR $\Psi(t_\nu)$

3.1. Since (see [5], p. 79)

$$\begin{aligned} \Xi(t) &= -\frac{1}{2\pi^{1/4}} \left( t^2 + \frac{1}{4} \right) \left| \Gamma \left( \frac{1}{4} + i \frac{t}{2} \right) \right| Z(t), \\ \left| \Gamma \left( \frac{1}{4} + i \frac{t}{2} \right) \right| &= 2^{1/4} \sqrt{2\pi} e^{-\frac{\pi}{4}} t^{-1/4} \{1 + \mathcal{O}(t^{-1})\}, \end{aligned}$$

then

$$(3.1) \quad \Xi(t) = - \left( \frac{\pi}{2} \right)^{1/4} \{1 + \mathcal{O}(t^{-1})\} e^{-\frac{\pi t}{4}} t^{7/4} Z(t).$$

Let

$$(3.2) \quad \Phi(T) = \int_T^\infty \Xi(t) dt = a\Phi_1(T) + \Phi_2(T)$$

where (see (2.18), (3.1))

$$(3.3) \quad \begin{aligned} \Phi_1(T) &= \int_T^\infty e^{-\alpha t} t^\beta Z(t) dt, \quad a = - \left( \frac{\pi}{2} \right)^{1/4}, \\ \Phi_2(T) &= \mathcal{O} \left( \int_T^\infty e^{-\alpha t} t^{\beta-1} |Z(t)| dt \right). \end{aligned}$$

Since  $Z(t) = \mathcal{O}(t^{1/4})$  then (comp. (2.9))

$$\Phi_2(T) = \mathcal{O}(e^{-\alpha T} T^{\beta-3/4}),$$

and consequently

$$(3.4) \quad \Phi(T) = a\Phi_1(T) + \mathcal{O}(e^{-\alpha T} T^{\beta-3/4}).$$

Hence, from (3.4) by (2.18), (3.3) the formula

$$(3.5) \quad \begin{aligned} \Psi(T) &= e^{\alpha T} T^{-\beta} \Phi(T) = \\ &= \frac{2a}{\alpha} \frac{1}{1 + \frac{1}{\alpha^2} \{\vartheta'(T)\}^2} \left[ \cos\{\vartheta(T)\} - \frac{1}{\alpha} \vartheta'(T) \sin\{\vartheta(T)\} \right] + \\ &+ \frac{2a}{\alpha} \sum_{2 \leq n \leq \sqrt{\frac{T}{2\pi}}} \frac{1}{\sqrt{n} (1 + \frac{1}{\alpha^2} \{\vartheta'(T) - \ln n\}^2)} \left[ \cos\{\vartheta(T) - T \ln n\} - \right. \\ &\left. - \frac{1}{\alpha} \{\vartheta'(T) - \ln n\} \sin\{\vartheta(T) - T \ln n\} \right] + \mathcal{O}(T^{-1/4}) \end{aligned}$$

follows for all sufficiently big  $T > 0$ , and  $\alpha = \frac{\pi}{4}$ ,  $\beta = \frac{7}{4}$ .

3.2. Since from (3.5) in the case  $T \rightarrow t$ ,  $t \in [T, T+H]$ ,  $H = o(T)$ , we have

$$(3.6) \quad \begin{aligned} \Psi(t) &= \frac{2a}{\alpha} \frac{1}{1 + \frac{1}{\alpha^2} \{\vartheta'(t)\}^2} \left[ \cos\{\vartheta(t)\} - \frac{1}{\alpha} \vartheta'(t) \sin\{\vartheta(t)\} \right] + \\ &+ \frac{2a}{\alpha} \sum_{2 \leq n \leq \sqrt{\frac{t}{2\pi}}} \frac{1}{\sqrt{n} (1 + \frac{1}{\alpha^2} \{\vartheta'(t) - \ln n\}^2)} \left[ \cos\{\vartheta(t) - t \ln n\} - \right. \\ &\left. - \frac{1}{\alpha^2} \{\vartheta'(t) - \ln n\} \sin\{\vartheta(t) - t \ln n\} \right] + \mathcal{O}(t^{-1/4}), \end{aligned}$$

and (see (2.2))

$$\begin{aligned}
 \frac{1}{1 + \frac{1}{\alpha^2}\{\vartheta'(t)\}^2} &= \frac{1}{1 + \frac{1}{\alpha^2}\{\vartheta'(T)\}^2} + \mathcal{O}\left\{\frac{\vartheta'(c)\vartheta''(c)}{(1 + \frac{1}{\alpha^2}\{\vartheta'(t)\}^2)^2}H\right\} = \\
 &= \frac{1}{1 + \frac{1}{\alpha^2}\{\vartheta'(T)\}^2} + \mathcal{O}\left(\frac{H}{T \ln^3 T}\right), \\
 \frac{1}{1 + \frac{1}{\alpha^2}\{\vartheta'(t) - \ln n\}^2} &= \frac{1}{1 + \frac{1}{\alpha^2}\{\vartheta'(T) - \ln n\}^2} + \mathcal{O}\left(\frac{H}{T \ln^3 T}\right), \\
 \vartheta'(t) - \ln n &= \vartheta'(T) - \ln n + \mathcal{O}\left(\frac{H}{T}\right), \\
 \sum_{\sqrt{\frac{T}{2\pi}} \leq n \leq \sqrt{\frac{T+H}{2\pi}}} \frac{1}{\sqrt{n}} &= \mathcal{O}\left(\frac{H}{T^{3/4}}\right)
 \end{aligned}$$

then from (3.6) the formula

$$\begin{aligned}
 \Psi(t) &= \frac{2a}{\alpha} \frac{1}{1 + \frac{1}{\alpha^2}\{\vartheta'(T)\}^2} \left[ \cos\{\vartheta(t)\} - \frac{1}{\alpha} \vartheta'(t) \sin\{\vartheta(t)\} \right] + \\
 (3.7) \quad &+ \frac{2a}{\alpha} \sum_{2 \leq n < P_0} \frac{1}{\sqrt{n} (1 + \frac{1}{\alpha^2}\{\vartheta'(T) - \ln n\}^2)} \left[ \cos\{\vartheta(t) - t \ln n\} - \right. \\
 &\left. - \frac{1}{\alpha} \{\vartheta'(T) - \ln n\} \sin\{\vartheta(t) - t \ln n\} \right] + \mathcal{O}(t^{-1/4})
 \end{aligned}$$

follows for

$$t \in [T, T + H], \quad H = \mathcal{O}\left(\frac{\sqrt{T}}{\ln T}\right), \quad P_0 = \sqrt{\frac{T}{2\pi}}.$$

Hence, from (3.7) by (2.3) we obtain the following

**Lemma 2.**

$$\begin{aligned}
 \Psi(t_\nu) &= \frac{2a}{\alpha} \frac{(-1)^\nu}{1 + \frac{1}{\alpha^2}\{\vartheta'(T)\}^2} + \\
 (3.8) \quad &+ \frac{2a}{\alpha} \sum_{2 \leq n < P_0} \frac{1}{\sqrt{n} (1 + \frac{1}{\alpha^2}\{\vartheta'(T) - \ln n\}^2)} \left\{ (-1)^\nu \cos(t_\nu \ln n) + \right. \\
 &\left. + \frac{1}{\alpha} \{\vartheta'(T) - \ln n\} (-1)^\nu \sin(t_\nu \ln n) \right\} + \mathcal{O}(t^{-1/4}), \\
 t_\nu &\in [T, T + H], \quad H = \mathcal{O}\left(\frac{\sqrt{T}}{\ln T}\right).
 \end{aligned}$$

#### 4. PROOF OF THE THEOREM

4.1. Since (see [3], (23))

$$\sum_{T \leq t_\nu \leq T+H} 1 = \frac{1}{\pi} H \ln P_0 + \mathcal{O}\left(\frac{H^2}{T}\right)$$

then we obtain from (3.8) the following equalities

$$\begin{aligned}
 \sum_{T \leq t_\nu \leq T+H} \Psi(t_\nu) &= w_1 + w_2 + \mathcal{O}(HT^{-1/4} \ln T), \\
 (4.1) \quad \sum_{T \leq t_\nu \leq T+H} (-1)^\nu \Psi(t_\nu) &= \frac{2a}{\pi\alpha} \frac{H \ln P_0}{1 + \frac{1}{\alpha^2} \{\vartheta'(T)\}^2} + w_3 + w_4 + \mathcal{O}(HT^{-1/4} \ln T), \\
 H &= \mathcal{O}(T^{1/4+\delta})
 \end{aligned}$$

for the main sums ( $0 < \delta$  is arbitrarily small) where

$$\begin{aligned}
 w_1 &= \sum_{2 \leq n < P_0} \frac{a_n}{\sqrt{n}} \sum_{T \leq t_\nu \leq T+H} (-1)^\nu \cos(t_\nu \ln n), \\
 w_2 &= \sum_{2 \leq n < P_0} \frac{a_n b_n}{\sqrt{n}} \sum_{T \leq t_\nu \leq T+H} (-1)^\nu \sin(t_\nu \ln n), \\
 (4.2) \quad w_3 &= \sum_{2 \leq n < P_0} \frac{a_n}{\sqrt{n}} \sum_{T \leq t_\nu \leq T+H} \cos(t_\nu \ln n), \\
 w_4 &= \sum_{2 \leq n < P_0} \frac{a_n b_n}{\sqrt{n}} \sum_{T \leq t_\nu \leq T+H} \sin(t_\nu \ln n),
 \end{aligned}$$

and (see (2.2))

$$\begin{aligned}
 (4.3) \quad a_n &= \frac{2a}{\alpha} \frac{1}{1 + \frac{1}{\alpha^2} \{\vartheta'(T) - \ln n\}^2}, \\
 b_n &= \frac{1}{\alpha} \{\vartheta'(T) - \ln n\}^2 = \frac{2}{\alpha} \ln \frac{P_0}{n} + \mathcal{O}\left(\frac{1}{T}\right),
 \end{aligned}$$

( $a_n$  is increasing and  $a_n b_n$  is decreasing).

4.2. Let us remind that we have proved (see [2], p. 38, (56), [3], (26)) for the sums

$$\begin{aligned}
 \bar{w}_1 &= \sum_{2 \leq n < m} \frac{1}{\sqrt{n}} \sum_{T \leq t_\nu \leq T+H} (-1)^\nu \cos(t_\nu \ln n), \\
 \bar{w}_2 &= \sum_{2 \leq n < m} \frac{1}{\sqrt{n}} \sum_{T \leq t_\nu \leq T+H} (-1)^\nu \sin(t_\nu \ln n), \\
 \bar{w}_3 &= \sum_{2 \leq n < m} \frac{1}{\sqrt{n}} \sum_{T \leq t_\nu \leq T+H} \cos(t_\nu \ln n), \\
 \bar{w}_4 &= \sum_{2 \leq n < m} \frac{1}{\sqrt{n}} \sum_{T \leq t_\nu \leq T+H} \sin(t_\nu \ln n), \quad m < P_0,
 \end{aligned}$$

where  $\sin(t_\nu \ln n) = \cos(t_\nu \ln n - \frac{\pi}{2})$  the following estimates

$$(4.4) \quad \bar{w}_1, \bar{w}_2 = \mathcal{O}(T^{1/6+\epsilon/4}),$$

(see [3], p. 97, (2)), and

$$(4.5) \quad \bar{w}_3, \bar{w}_4 = \mathcal{O}(T^{1/6+\epsilon/4}),$$

(see [3], p. 48, (26) and p. 98, (5);  $\Delta = \frac{1}{6}$ ). Since  $\{a_n\}$  and  $\{a_n b_n\}$  are monotonic sequences and (see (4.3))

$$a_n = \mathcal{O}(1), \quad a_n b_n = \mathcal{O}(\ln P_0) = \mathcal{O}(\ln T)$$



we obtain from (4.2), using Abel's transformation and (4.4), (4.5), the following estimates

$$(4.6) \quad w_1, w_2, w_3, w_4 = \mathcal{O}(T^{1/6+\epsilon/2}).$$

4.3. Thus, from (4.1) by (4.6) we obtain

$$(4.7) \quad \sum_{T \leq t_\nu \leq T+H} \Psi(t_\nu) = \mathcal{O}(T^{1/6+\epsilon/2}),$$

$$\sum_{T \leq t_\nu \leq T+H} (-1)^\nu \Psi(t_\nu) = \frac{2a}{\pi\alpha} \frac{H \ln P_0}{1 + \frac{1}{\alpha^2} \{\vartheta'(T)\}^2} + \mathcal{O}(T^{1/6+\epsilon/2})$$

for  $H = \mathcal{O}(T^{1/4+\delta})$ . Since (see (2.2))

$$1 + \frac{1}{\alpha^2} \{\vartheta'(T)\}^2 \sim \frac{4}{\alpha^2} \ln^2 P_0,$$

we obtain from (4.7) the following asymptotic formulae

$$(4.8) \quad \sum_{T \leq t_{2\nu} \leq T+\bar{H}} \Psi(t_{2\nu}) \sim \frac{a\alpha}{\pi} \frac{\bar{H}}{\ln P_0},$$

$$\sum_{T \leq t_{2\nu+1} \leq T+\bar{H}} \Psi(t_{2\nu+1}) \sim -\frac{a\alpha}{\pi} \frac{\bar{H}}{\ln P_0}, \quad \bar{H} = \frac{1}{3} T^{1/6+\epsilon}.$$

Hence, it follows from (4.8) that there is a zero  $\omega$  of the odd order of the function  $\Psi(t)$ ,  $t \in [T, T + \bar{H}]$ , i.e. by (3.5),  $\omega$  is the zero of the odd order of the function

$$\Phi(t) = \int_t^\infty \Psi(\tau) d\tau.$$

I would like to thank Michal Demetrian for helping me with the electronic version of this work.

#### REFERENCES

- [1] G.H. Hardy, 'Sur les zeros de la fonction  $\zeta(s)$  de Riemann', Compt. Rendus Acad. Sci., 158 (1914), 1012-1014.
- [2] J. Moser, 'On one sum in the theory of the Riemann's zeta-function', Acta Arith., 31 (1976), 31-43.
- [3] J. Moser, 'On one theorem of Hardy-Littlewood in the theory of the Riemann's zeta-function', Acta Arith., 31 (1976), 45-51; 40 (1981), 97-107.
- [4] J. Moser, 'Jacob's ladders and the tangent law for short parts of the Hardy-Littlewood integral', (2009), arXiv: 0906 0659.
- [5] E.C. Titchmarsh, 'The theory of the Riemann zeta-function', Clarendon Press, Oxford, 1951.

DEPARTMENT OF MATHEMATICAL ANALYSIS AND NUMERICAL MATHEMATICS, COMENIUS UNIVERSITY, MLYNSKA DOLINA M105, 842 48 BRATISLAVA, SLOVAKIA

*E-mail address:* jan.moser@fmph.uniba.sk